Calculus 2 Study Guide

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1 Review

1.1 A few important algebraic formulas

Dividing by a number is the same as multiplying by its reciprocal:

$$\frac{a}{b} = a * \frac{1}{b}$$

Examples:

$$\frac{3}{5} = 3 * \frac{1}{5}$$

$$\frac{x}{\frac{1}{4}} \;=\; x * \frac{4}{1} \;=\; x * 4 \;=\; 4x$$

Difference of Squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Examples:

$$x^2 - 16 = (x+4)(x-4)$$

$$x - 16 = (\sqrt{x}+4)(\sqrt{x}-4)$$

Note: This is not the same as $(a-b)^2$ which is equal to $(a-b)(a-b) = a^2 - 2ab + b^2$.

2 Limits

2.1 One-sided Limits

Left sided limit: $\lim\limits_{x \to a^-} f(x) = b$ **Right sided limit:** $\lim\limits_{x \to a^+} f(x) = b$

Existance of two-sided limits from one-sided limits

If $\lim\limits_{x \to a^-} f(x) = b$ and $\lim\limits_{x \to a^+} f(x) = b$, then $\lim\limits_{x \to a} f(x) = b$.

If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $\lim\limits_{x \to a} f(x)$ DNE (Does Not Exist).

If $\lim\limits_{x \to a^-} f(x)$ DNE, then $\lim\limits_{x \to a} f(x)$ DNE.

If $\lim\limits_{x \to a^+} f(x)$ DNE, then $\lim\limits_{x \to a} f(x)$ DNE.

2.2 Limit Properties

In the below properites a, c, and n are all constants.

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a$$

$$\lim_{x\to a}\Big(cf(x)\Big) = c \cdot \lim_{x\to a} f(x) \qquad \lim_{x\to a}\Big(f(x) \pm g(x)\Big) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\Big(f(x)\cdot g(x)\Big) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \qquad \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\text{ if } \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a}\Big(f(x)\Big)^n = \Big(\lim_{x\to a} f(x)\Big)^n \qquad \lim_{x\to a}\Big(\sqrt[n]{f(x)}\Big) = \sqrt[n]{\lim_{x\to a} f(x)}$$

3 Integrals - Definition and Properties

3.1 Definition of a Definite Integral

If f is a function that's defined for $a \leq x \leq b$, and we split the interval $[a, b]$ into n pieces of equal width $\Delta x = \frac{b-a}{n}$. The we say that $x_0, x_1, x_2, ..., x_n$ are the endpoints of these smaller pieces where $x_0 = a$ and $x_n = b$. So $x_1^*, x_2^*, ..., x_n^*$ would be sample points within each of the split pieces of the overall interval, so x_i^* is in the subinterval $[x_{i-1}, x_i]$. Then we can define the integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Remember, this represents the sum of the area of a bunch of infinitely thin rectangles. The hieght of each rectangle is the height of the function at each little piece ($f(x_i^*)$). And the width of each rectangle is the distance you go over to the side to get to the next rectangle (Δx).

3.2 Left Endpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_{i-1} = a + (i-1)\Delta x$$

3.3 Right Endpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x$$

3.4 Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\big[f(\bar{x}_1) + ... + f(\bar{x}_n)\big]$$

where $\quad \Delta x = \frac{b-a}{n} \quad$ and $\quad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of\ [x_{i-1}, x_i]$

3.5 Definite Integral Properties

$\int_b^a f(x)dx = -\int_a^b f(x)dx$

$\int_a^a f(x) = 0$

$\int_a^b c\ dx = c(b-a)$, where c is a constant

$\int_a^b \big[f(x) + g(x)\big]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

$\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant

$\int_a^b \big[f(x) - g(x)\big]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

3.6 Fundamental Theorem of Calculus

3.6.1 Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$, then the function g is defined by

$$g(x) = \int_a^x f(t)\ dt \quad a \leq x \leq b$$

is an antiderivative of f, meaning $g'(x) = f(x)$ for $a < x < b$.

3.6.2 Fundamental Theorem of Calculus Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is any antiderivative of f, menaing $F' = f$.

4 Integrals - Techniques for Evaluating

4.1 u-substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)g'(x)\ dx = \int f(u)\ du$$

When using u-substitution, you usually want to look for some piece of your function that has its derivative somewhere else in the function. Make that piece equal u. Then when you find du (the derivative of u), that will be somewhere in the function as well.

We can also use u-substitution for definite integrals. If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\big(g(x)\big)g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

All this means is that you have to convert your bounds of the integral to be in terms of u as well. Just plug the bounds of the original integral into the piece you have called u to convert them to be in terms of u.

4.2 Integrals of Symetric Functions

As long is f is continuous on $[-a, a]$.

1. If f is even $\big[\text{which means}: \ f(-x) = f(x)\big]$, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$.
2. If f is odd $\big[\text{which means}: \ f(-x) = -f(x)\big]$, then $\int_{-a}^{a} f(x)\ dx = 0$.

4.3 Integration by Parts

$$\begin{aligned} \int f(x)g'(x)\ dx &= f(x)g(x) - \int g(x)f'(x)\ dx \\ \int u\ dv &= uv - \int v\ du \end{aligned}$$

To apply this formula, first decide which piece of the function will be u and which will be dv. Then find du (derivative of the piece you called u) and find v (the antiderivative of dv). Then plug all of these pieces into the formula and you should be left with a simpler integral.

You can also use these formulas similarly with definite integrals.

$$\int_a^b f(x)g'(x)\ dx \ = \ f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\ dx$$

$$\int_a^b u\ dv \ = \ uv\Big|_a^b - \int_a^b v\ du$$

4.4 Improper Integrals

4.4.1 Infinite Intervals

(a) If $\int_a^t f(x)\ dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

assuming this limit exists.

(b) If $\int_t^b f(x)\ dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$$

provided this limit exists as a finite number.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are **convergent** is the limit exists, and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^a f(x)\ dx$ are convergent, then for any real number a

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^a f(x)\ dx + \int_a^\infty f(x)\ dx$$

(d) $\int_1^\infty \frac{1}{x^p}\ dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

4.5 Discontinuous Integrands

(a) If f is continuous on $[a,\ b)$ and is discontinuous at b, then

$$\int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx$$

if this limit exists as a finite number.

(b) If f is continuous on $(a,\ b]$ and is discontinuous at a, then

$$\int_a^b f(x)\ dx = \lim_{t\to a^+} \int_t^b f(x)\ dx$$

if this limit exists as a finite number.

The improper integral $\int_a^b f(x)\ dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$ are convergent, then we define

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

5 Applications of Integration

The area A of the region bounded between the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, and $x = b$ where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a,\ b]$.

$$A = \int_a^b \big[f(x) - g(x)\big]\, dx$$

5.1 Solids of Revolution

Washer or Disk Method: $\pi \int_a^b$ (outer radius)^2-(inner radius)^2 dx

You'll need to figure out the inner and outer radius by considering the distance between the radii and the axis of rotation.

Cylindrical Shell Method: $\int 2\pi rh\ dx$

5.2 Arc Length Formula

If a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b, then its length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

If you are instead given a curve with an equation in the form $y = f(x)$ and $a \leq x \leq b$, then we can find arc length using

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If you are instead given a curve with an equation in the form $x = f(y)$ and $a \leq y \leq b$, then we can find arc length using

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$$

5.3 Average Value of a Function

The average value of a function f on the interval $[a,\ b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

5.4 Work

Work = Force * Distance

5.5 Center of Mass

The center of mass of a plate is located at the point $(\bar{x},\ \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\big[f(x)\big]^2\ dx$$

and A is the area of the plate.

6 Differential Equations

6.1 Euler's Method

Euler's Method can be used to approximate values for the solution to an initial-value problem $y' = F(x,\ y),\ y(x_0) = y_0$, with a step size h, at $x_n = x_{n-1} + h$, by

$$y_n = y_{n-1} + hF(x_{n-1},\ y_{n-1}) \qquad n = 1,\ 2,\ 3,\ ...$$

6.2 Separable Differential Equations

A separable differential equation is one where you can separate all of the x terms and dx on one side of the equation and all the y terms and dy on the other side. They will be in the form $\frac{dy}{dx} = g(x)f(y)$. They can be solved with the following steps:

$$\frac{dy}{dx} = g(x)f(y)$$

$$\frac{dy}{f(y)} = g(x)\ dx$$

$$\int \frac{1}{f(y)}\ dy = \int g(x)\ dx$$

Then integrate and solve for y.

Similarly, we can also have a differential equation in the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ and can be solved with a similar process:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$h(y)\ dy = g(x)\ dx$$

$$\int h(y)\ dy = \int g(x)\ dx$$

Then integrate and solve for y.

6.3 Exponential Growth and Decay

The solution to the initial-value problem for some constant k

$$\frac{dy}{dt} = ky \qquad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt}$$

Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_s)$$

where $T(t)$ is the temperature of the object at time t, and T_s is the temperature of the surroundings.

Continuously Compunded Interest:

$$A(t) = A_0 e^{rt}$$

where $A(t)$ is the value of the investment at time t, A_0 is the value of the investment at time $t = 0$, and r is the annual interest rate of the continuous compound interest (as a decimal, not a percent).